Differential Topology — May 2024

End-Semester Exam — 100 Points

**Notes.**

(a) $\mathbb{R}$ = real numbers

(b) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified. You may however, use the following fact without proof: For any point $p$ in the $k$-sphere $S^k$, there is a diffeomorphism between $S^k \setminus \{p\}$ and the Euclidean space $\mathbb{R}^k$.

(c) There are a total of **115** points in this paper. You will be awarded a maximum of **100** points.

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1. [25 Points] Let $f\colon X \to Y$ be a proper map of boundaryless manifolds of the same dimension. Suppose $y \in f(X)$ is a regular value of $f$. Show that $f^{-1}(y)$ is a finite set, say $\{x_1, \dots, x_n\}$. Moreover, prove that there exists a neighbourhood $U$ of $y$ in $Y$ such that $f^{-1}U$ is a disjoint union $V_1 \cup \cdots \cup V_n$ where each $V_i$ is an open neighbourhood of $x_i$ in $X$ and $f$ maps each $V_i$ diffeomorphically onto $U$.

2. [15 Points] Find all values of $a > 0$ for which the hyperboloid $x^2 + y^2 = z^2 + 1$ intersects the sphere $x^2 + y^2 + z^2 = a$ transversally. Draw pictures of the two surfaces for typical values of $a$ according to when the intersection is empty or when it is non-transversal or when it is non-empty and tranversal respectively.

3. [5 Points] Give an example of a non-compact manifold $X$ of dimension at least 2 such that its boundary $\partial X$ is nonempty and compact.

4. [10 + 20 = 30 Points] Let $X$ be a boundaryless $k$-manifold in $\mathbb{R}^M$.
   (i) Define the tangent bundle $T(X)$ and the normal bundle $N(X)$. As manifolds, what are their dimensions ?
   (ii) If $X = S^1 \subset \mathbb{R}^2$ (the unit circle in the plane), then prove that both $T(X)$ and $N(X)$ are diffeomorphic to $X \times \mathbb{R}$.

5. [10 Points] Let $f\colon X \to Y$ be a map of boundaryless manifolds of the same dimension with $X$ compact and $Y$ connected, non-compact. Prove that $\deg_2(f) = 0$.

6. [10 + 10 + 10 = 30 Points] Let $X, Z$ be compact boundaryless manifolds of complementary dimension in a boundaryless manifold $Y$.
   (i) If $Y = \mathbb{R}^n$, prove that $I_2(X, Z) = 0$.
   (ii) If $Y = S^n$ (the unit sphere in $\mathbb{R}^{n+1}$) and both $X, Z$ have positive dimension, then prove that $I_2(X, Z) = 0$.
   (iii) Give an example where $X, Z$ have positive dimension and $I_2(X, Z) = 1$.